



Information Processing Capacity of Dynamical Systems

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Many dynamical systems, both natural and artificial, are stimulated by time dependent external signals, somehow processing the information contained therein. We demonstrate how to quantify the different modes in which information can be processed by such systems and combine them to define the computational capacity of a dynamical system. This is bounded by the number of linearly independent state variables of the dynamical system, equaling it if the system obeys the *fading memory* condition. It can be interpreted as the total number of linearly independent functions of its stimuli the system can compute. Our theory combines concepts from machine learning (*reservoir computing*), system modeling, stochastic processes, and functional analysis. We illustrate our theory by numerical simulations for the logistic map, a recurrent neural network, and a two-dimensional reaction diffusion system, uncovering universal trade-offs between the non-linearity of the computation and the system's short-term memory.

Many dynamical systems, both natural and artificial, are driven by external input signals and can be seen as performing non-trivial, real-time computations of these signals. The aim of this work is to study the online information processing that takes place within these dynamical systems. The importance of this question is underlined by the multitude of examples from nature and engineering which fall into this category. These include the activation patterns of biological neural circuits to sensory input streams¹, the dynamics of many classes of artificial neural network models used in artificial intelligence (including the whole field of “reservoir computing”^{2–5} as well as its recent implementation using time delay systems^{6–8}), and systems of interacting chemicals, such as those found within a cell, that display intra-cellular control mechanisms in response to external stimuli⁹. Recent insight in robotics has demonstrated that the control of animal and robot locomotion can greatly be simplified by exploiting the physical body's response to its environment^{10,11}. Additional systems to which our theory could apply are population dynamics responding to, e.g., changes in food supply, ecosystems responding to external signals such as global climate changes, and the spread of information in social network graphs. All of these exemplary systems display completely different physical implementations.

We present a general framework that allows to compare the computational properties of a broad class of dynamical systems. We are able to characterize the information processing that takes place within these systems in a way which is independent of their physical realization, using a quantitative measure. It is normalized appropriately such that completely different systems can be compared and it allows us to characterize different computational regimes (linear vs. non-linear; long vs. short memory).

Some initial steps in this direction are provided by the linear memory capacity introduced in¹² (later theoretically extended to linear systems in discrete time¹³ and continuous time systems¹⁴, and using Fischer information¹⁵). Although it has been argued that short-term memory of a neural system is crucial for the brain to perform useful computation on sensory input streams^{15,16}, it is our belief that the general paradigm underlying the brain's computation cannot solely rely on maximizing linear memory. The neural circuits have to compute complex non-linear and spatio-temporal functions of the inputs. Prior models focusing on linear memory capacity in essence assumed the dynamic system to only implement memory, while the complex non-linear mapping is off-loaded to an unspecified read-out mechanism. We propose to endow the dynamic system with all the required computational capacity, and use only simple linear read-out functions. The capacity measures we introduce are therefore of great interest since they quantify all the information processing that takes place within the dynamical system, and don't introduce an artificial separation between linear and non-linear information processing.

One of the startling results of this work with potentially far reaching consequences is that all dynamical systems, provided they obey the condition of having a fading memory¹⁷ and have linearly independent internal variables, have in principle the same total normalized capacity to process information. The ability to carry out

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useful information processing is therefore an almost universal characteristic of dynamical systems. This result provides theoretical justification for the widely used paradigm of reservoir computing with a linear readout layer²⁻⁴; indeed it confirms that such systems can be universal for computation of time invariant functions with fading memory.

We illustrate the versatility of our approach by using numerical simulations on three very different dynamical systems: the logistic map, a recurrent neural circuit, and a two-dimensional reaction diffusion system. These examples allow us to uncover the (potentially universal) trade-off between memory depth and non-linear computations performed by the dynamic system itself. We also discuss how the influence of noise decreases the computation capacity of a dynamical system.

Results

Dynamical systems. The dynamical systems we consider are observed in discrete time $t \in \mathbb{Z}$. They are described by internal variables $x_j(t)$, $j \in J$, being the set of indices of all internal variables. The set J may be finite or (enumerably or continuously) infinite.

The internal variables are driven by external signals $u(t) \in U^K$ which can have arbitrary dimensionality. Typical examples for the set U include a discrete set, or subsets of \mathbb{R}^K such as the unit interval or the whole real line \mathbb{R} . When studying a dynamical system, we do not necessarily have access to all the dynamical variables $x(t)$. This corresponds to experimental situations where many variables may not be measurable, and is also the case if the number of dynamical variables $|J|$ is infinite. We therefore assume that we can only access the instantaneous state of the system through a finite number N of the internal variables: $x_i(t)$, $i = 1, \dots, N$. Note that we use the term internal variables throughout the paper for easy of notation. The theory however also holds if we cannot directly observe the internal states, but only have access to them via N observation functions. An important special case is when the dynamical system comprises a finite number $|J| = N$ of dynamical variables and we have access to all of them.

The system evolves according to an evolution equation of the form:

$$x_j(t) = T_j(x(t-1), u(t)), \quad (1)$$

with T_j the mapping from input and previous state to the j 'th current state. We denote a dynamical system with the evolution law eq. (1) by the capital letter X .

In order to evaluate the performance of the system in a standardized way and to make analytical derivations tractable, we will take the inputs $u(t)$ to be independent and identically drawn from some probability distribution $p(u)$. It should be noted that in many, if not all, real world situations the inputs are not i.i.d. Most of the theory should also hold for the non-i.i.d. case, but this significantly complicates the construction of a basis for the Hilbert space defined in Definition 5. However our aim is to characterize the dynamical system itself. By taking the inputs to be i.i.d. we eliminate as much as possible any observed structure originating from the inputs, so that any measured structure will be due to the dynamical system itself. The distribution $p(u)$ can be chosen according to what is most relevant for the dynamical system under study.

In order to study this dynamical system experimentally, we proceed as follows. First the system is initialized in an arbitrary initial state at a time $-T'$, and is driven for T' time steps with a sequence of i.i.d. inputs drawn from the distribution $p(u)$. This to ensure that potential initial transients have died out. Next, we drive the system for T timesteps with inputs from the same distribution. During this later phase we record both the values of the inputs $u(t)$ and of the state variables $x(t)$ which will be used to get insight into how information is processed by the dynamical system.

Capacity for reconstructing a function of the inputs. We will denote by $u^{-h}(t) = (u(t-h+1), \dots, u(t))$ the sequence of h previous inputs up to time t . It follows that $u^{-\infty}(t) = (\dots, u(t-2), u(t-1), u(t))$ is the left infinite sequence of all previous inputs up to time t . We will often consider functions of a finite number h , or an infinite number of time-steps (the latter corresponding to the limit $T, T' \rightarrow \infty$). We often wish to refer to a sequence of inputs without referring to the time at which they act, in which case we denote by $u^{-h} = (u_{-h+1}, \dots, u_0) \in U^h$ a sequence of h inputs and by $u^{-\infty} = (\dots, u_{-2}, u_{-1}, u_0) \in U^\infty$ a left infinite sequence of inputs.

Consider a function on sequences of h inputs

$$z : U^h \rightarrow \mathbb{R} : u^{-h} \rightarrow z[u^{-h}]. \quad (2)$$

Given the recorded inputs $u(t)$, it induces a time dependent function $z(t) = z(u^{-h}(t))$.

A central quantity in our analysis will be the *capacity* to reconstruct the function z from the state of a dynamical system using a linear estimator. We introduce this notion as follows. Consider a linear estimator constructed from the observed data

$$\hat{z}(t) = \sum_{i=1}^N W_i x_i(t). \quad (3)$$

We measure the quality of the estimator by the *Mean Square Error* (MSE):

$$MSE_T[\hat{z}] = \frac{1}{T} \sum_{t=1}^T (\hat{z}(t) - z(t))^2. \quad (4)$$

where here and throughout we denote by a subscript T quantities that are generated from the measured data. We are interested in the estimator which minimizes the MSE.

Definition 1. The *Capacity* for the dynamical system X to reconstruct the function z is

$$C_T[X, z] = 1 - \frac{\min_{W_i} MSE_T[\hat{z}]}{\langle z^2 \rangle_T} \quad (5)$$

where $\langle z^2 \rangle_T = \frac{1}{T} \sum_{t=1}^T z(t)^2$.

The notion of capacity can also be interpreted from the point of view of computation. In this approach, one wishes to use the dynamical system to compute the function $z(u^{-h})$, and to this end one uses, following the ideas introduced in reservoir computing²⁻⁴, the simple linear readout eq. (3). The capacity $C[X, z]$ measures how successful the dynamical system is at computing z . Note that this capacity is essentially the same measure that was introduced in¹² when $z(t) = u(t-l)$ is a linear function of one of the past inputs. The main difference with the notation of² is the treatment of a possible bias term in eq. (3). This corresponds to writing $\hat{z}(t) = \sum_{i=1}^N W_i x_i(t) + W_{N+1}$ and is equivalent to introducing an additional observation function $x_{N+1}(t) = 1$ which is time independent. In what follows we do not introduce such a bias term, as this makes the statement of the results simpler. We discuss the relation between the two approaches in the supplementary material.

We note several properties of the capacity:

Proposition 2. For any function $z(t)$, and any set of recorded data $x(t)$, $t = 1, \dots, T$ the capacity can be expressed as

$$C_T[X, z] = \frac{\sum_{ij} \langle z x_i \rangle_T \langle x_i x_j \rangle_T^{-1} \langle x_j z \rangle_T}{\langle z^2 \rangle_T}, \quad (6)$$

where $\langle v \rangle_T = \frac{1}{T} \sum_{t=1}^T v(t)$ denotes the time average of a set of data, and $\langle x_i x_j \rangle_T^{-1}$ denotes the inverse of the matrix $\langle x_i x_j \rangle_T$. If $\langle x_i x_j \rangle_T$ does not have maximum rank, then $\langle x_i x_j \rangle_T^{-1}$ is zero on the kernel of $\langle x_i x_j \rangle_T$ and equal to its inverse on the complementary subspace and



Proposition 3. For any function z , and any set of recorded data, the capacity is normalized according to:

$$0 \leq C_T[X, z] \leq 1, \quad (7)$$

where $C_T[X, z] = 0$ if the dynamical system is unable to even partially reconstruct the function z , and $C_T = 1$ if perfect reconstruction of z is possible (that is if there is a linear combination of the observation functions $x_i(t)$ that equals $z(t)$).

Note that $C_T[X, z]$ depends on a single set of recorded data. The computed value of the capacity is therefore affected by statistical fluctuations: if one repeats the experiment one will get a slightly different value. This must be taken into account, particularly when interpreting very small values of the capacity C_T . We discuss this issue in the supplementary material.

The capacity is based on simple linear estimators $\hat{z}(t) = \sum_{i=1}^N W_i x_i(t)$. For this reason the capacity characterizes properties of the dynamical system itself, rather than the properties of the estimator. Consider for instance the case where $z[u^{-h}] = u_{-3}$ is the input 3 time steps in the past. Because the estimator is a linear combination of the states $x_i(t)$ of the dynamical system x_i at time t , a nonzero capacity for this function indicates that the dynamical system can store information in a linear subspace of $x(t)$ about the previous inputs for at least 3 time steps. Consider now the case where $z[u^{-h}] = u_{-2}u_{-3}$ is a nonlinear function of the previous inputs (in this case the product of the inputs 2 and 3 time steps in the past). Then a nonzero capacity for this function indicates that the dynamical system both has memory, and is capable of nonlinear transformations on the inputs, since the dynamical system is the only place where a nonlinear transformation can occur. We therefore say that if $C_T(X, z) > 0$, the dynamical system X can, with capacity $C(X, z)$, approximate the function z .

In summary, by studying the capacity for different functions we can learn about the information processing capability of the dynamical system itself. Because the capacity is normalized to 1, we can compare the capacity for reconstructing the function z for different dynamical systems.

Maximum capacity of a dynamical system. Consider two functions z and z' , and the associated capacities $C_T[X, z]$ and $C_T[X, z']$. If the functions z and z' are close to each other, then we expect the capacities to also be close to each other. On the other hand if z and z' are sufficiently distinct, then the two capacities will be giving us independent information about the information processing capability of the dynamical system. To further exploit the notion of capacity for reconstructing functions z , we therefore need to introduce a distance on the space of functions $z : U^h \rightarrow \mathbb{R}$. This distance should tell us when different capacities are giving us independent information about the dynamical system.

To introduce this distance, we note that the probability distribution $p(u)$ on the space of inputs, and associated notion of integrability, provide us with an inner product, distance, and norm on the space of functions $z : U^h \rightarrow \mathbb{R} : \langle z, z' \rangle_{U^h} = E_{U^h}[zz']$, $d(z, z') = E_{U^h}[(z - z')^2]$, $\|z\|_{U^h}^2 = \langle z, z \rangle_{U^h}$, where $E_{U^h}[\cdot]$, is the expectation taken with respect to the measure $p(u)$ on U^h . We define the Hilbert space \mathcal{H}_{U^h} as the space of functions $z : U^h \rightarrow \mathbb{R}$ with bounded norm $\langle z, z \rangle_{U^h} < \infty$ (this is technically known as the weighted L^2 space). In the following we suppose that \mathcal{H}_U , and hence \mathcal{H}_{U^h} , is separable, i.e., that it has a finite or denumerable basis.

The following theorem is one of our main results. It shows that if functions z and z' in \mathcal{H}_{U^h} are orthogonal, then the corresponding capacities $C_T[X, z]$ and $C_T[X, z']$ measure independent properties of the dynamical system, and furthermore the sum of the capacities for orthogonal state variables cannot exceed the number N of output functions of the dynamical system.

Theorem 4. Consider a dynamical system as described above with N output functions $x_i(t)$ and choose a positive integer $h \in \mathbb{N}$. Consider

any finite set $Y_L = \{y_1, \dots, y_L\}$ of size $|Y_L| = L$ of orthogonal functions in \mathcal{H}_{U^h} , $y_l \in \mathcal{H}_{U^h}, \langle y_l, y_{l'} \rangle_{U^h} = |y_l|^2 \delta_{ll'}, l, l' = 1, \dots, L$. We furthermore require that the fourth moment of the y_l is finite $\langle y_l^4 \rangle_{U^h} < \infty$. Then, in the limit of an infinite data set $T \rightarrow \infty$, the sum of the capacities for these functions is bounded by the number N of output functions (independently of h , of the set Y_L , or of its size $|Y_L| = L$):

$$\lim_{T \rightarrow \infty} \sum_{l=1}^L C_T[x, y_l] \leq N. \quad (8)$$

The fact that the right hand side in eq. (8) depends only on the number of output functions N implies that we can compare sums of capacities, for the same sets of functions Y_L , but for completely different dynamical systems.

Fading Memory. In order to characterize when equality is attained in Theorem 4 we introduce the notion of fading memory Hilbert space and fading memory dynamical system:

Definition 5. Hilbert space of fading memory functions. The Hilbert space \mathcal{H}_{U^∞} of fading memory functions is defined as the limit of Cauchy sequences in \mathcal{H}_{U^h} , as h increases, as follows:

1. If $x \in \mathcal{H}_{U^h}$, then $x \in \mathcal{H}_{U^\infty}$, and $\langle x, x \rangle_{U^\infty} = \langle x, x \rangle_{U^h}$.
2. Consider a sequence $x_h \in \mathcal{H}_{U^h}$, $h \in \mathbb{N}$, then the limit $\lim_{h \rightarrow \infty} x_h$ exists and is in \mathcal{H}_{U^∞} if for all $\epsilon > 0$, there exists $h_0 \in \mathbb{N}$, such that for all $h, h' > h_0$, $\|x_h - x_{h'}\|_{U^{\max(h, h')}}^2 < \epsilon$.
3. Conversely, all $x \in \mathcal{H}_{U^\infty}$ are the limit of a sequence of the type given in 2.
4. The scalar product of $x, x' \in \mathcal{H}_{U^\infty}$ is given by $\langle x', x \rangle_{U^\infty} = \lim_{h, h' \rightarrow \infty} \langle x'_h, x_h \rangle$, where $x_h \rightarrow x$ and $x'_h \rightarrow x'$ are any two Cauchy sequences that converge to x and x' according to 2) above.

In the supplementary information we prove that \mathcal{H}_{U^∞} indeed constitutes a Hilbert space, and that it has a denumerable basis if \mathcal{H}_U is separable.

Definition 6. Fading Memory Dynamical System¹⁷. Consider a dynamical system given by the recurrence $x(t) = T(x(t-1), u(t))$ with some initial conditions $x(-T')$, and suppose that one has access to a finite number of internal variables $x_i(t)$, $i = 1, \dots, N$. This dynamical system has fading memory if, for all $\epsilon > 0$, there exists a positive integer $h_0 \in \mathbb{N}$, such that for all $h > h_0$, for all initial conditions x_0 , and for all sufficiently long initialization time $T' > h$, the variables $x_i(t)$ at any time $t \geq 0$ are well approximated by the functions x_i^h :

$$E_{U^{t+T'}}(x_i(t) - x_i^h[u^{-h}(t)])^2 < \epsilon, \quad (9)$$

where the expectation is taken over the $t + T'$ previous inputs.

When the dynamical system has fading memory, its output functions $x_i(t)$ can be identified with vectors in the Hilbert space of fading memory functions. Indeed the convergence condition eq. (9) in the definition of fading memory dynamical systems is precisely the condition that the functions x_i^h have a limit in \mathcal{H}_{U^∞} . We can thus identify $\lim_{T' \rightarrow \infty} x_i(t) = x_i[u^{-\infty}(t)] \in \mathcal{H}_{U^\infty}$, where $x_i[u^{-\infty}] = \lim_{h \rightarrow \infty} x_i^h[u^{-h}]$.

The following result states that if the dynamical system has fading memory, and if the dynamical variables x_i , $i = 1, \dots, N$, are linearly independent (their correlation matrix has full rank), and if the function y_l constitute a complete orthonormal basis of the fading memory Hilbert space \mathcal{H}_{U^∞} , then one has equality in Theorem 4:

Theorem 7. Consider a dynamical system with fading memory as described above with N accessible variables $x_i(t)$. Because the dynamical system has fading memory, we can identify the output functions with functions $x_i(u^{-\infty})$ in \mathcal{H}_{U^∞} . Consider an increasing family of orthogonal functions in $\mathcal{H}_{U^\infty} : Y_L = \{y_1, \dots, y_L\}$ with $Y_L \subseteq Y_{L'}$ if $L' \geq L$ and $y_l \in \mathcal{H}_{U^\infty}, \langle y_l, y_{l'} \rangle_{U^\infty} = |y_l|^2 \delta_{ll'}, l, l' = 1, \dots, L$, such that in the limit $L \rightarrow \infty$, the sets Y_L tends towards a complete orthogonal set of functions in \mathcal{H}_{U^∞} . Suppose that the readout functions $x_i(u^{-\infty})$ and the basis functions $y_l(u^{-\infty})$ have finite fourth order moment:



$\langle x_i^4 \rangle_{U^\infty} < \infty$, $\langle y_i^4 \rangle_{U^\infty} < \infty$. Consider the limit of an infinite data set $T \rightarrow \infty$ and infinite initialization time $T' \rightarrow \infty$. Suppose the correlation matrix $R_{i'i'} = \lim_{T, T' \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_i(t)x_{i'}(t)$ has rank N . Then the sum of the capacities for the sets Y_L tends towards the number N of output functions:

$$Y_L \rightarrow \lim_{\text{complete set}} \left[\lim_{T, T' \rightarrow \infty} \sum_{l=1}^L C_T[x, y_l] \right] = N. \quad (10)$$

This result tells us that under the non-strict conditions of Theorem 7 (fading memory and linearly independent internal variables), all dynamical systems driven by an external input and comprising N linearly independent internal variables have the same ability to carry out computation on the inputs, at least as measured using the capacities. Some systems may carry out more interesting computations than others, but at the fundamental level the amount of computation carried out by the different systems is equivalent.

Connections with other areas. There is a close connection between the sum of capacities for orthogonal functions $\sum_l C_T[X, y_l]$ and system modeling, and in particular Wiener series representation of dynamical systems^{18,19}. Indeed suppose that one wishes to model the dynamical system with variables x_i (which for simplicity of the argument we suppose to belong to \mathcal{H}_{U^∞}) by linear combinations of the finite set of orthonormal functions $y_l \in \mathcal{H}_{U^\infty}$, $l = 1, \dots, L$. The best such approximation is $\sum_{l=1}^L y_l \langle y_l, x_i \rangle_{U^\infty}$, and we can write: $x_i = \sum_{l=1}^L y_l \langle y_l, x_i \rangle_{U^\infty} + \Delta_i$ where the errors Δ_i are orthogonal to the y_l . We can then write the sum of the capacities as

$$\lim_{T \rightarrow \infty} \sum_{l=1}^L C_T[x, y_l] = N \left(1 - O \left(\frac{\|\Delta\|_{U^\infty}^2}{\|x\|_{U^\infty}^2} \right) \right).$$

That is, the extent to which the sum of the capacities for a finite set of orthogonal functions saturates the bound given in Theorem 7 is proportional to how well the dynamical system can be modeled by the linear combinations of these functions. This connection between the degree to which dynamical systems can be used for computation and system modeling has to our knowledge not been noted before.

Reservoir computing was one of the main inspirations for the present work, and is closely connected to it. We shed new light on it, and in particular on the question raised in² of characterizing the combinations of dynamical systems and output layer that have universal computing power on time invariant functions with fading memory. In² it was shown that the system has universal computing power if the dynamical system has the *point-wise separation property* (a weak constraint), and if the readout layer has the *approximation property* (a strong constraint). Here we consider simple readout layers which are linear functions of the internal states of the dynamical system, see eq. (3). Our work shows that the combination of a dynamical system and a linear output layer has universal computing power on time invariant functions with fading memory if, in the limit $N \rightarrow \infty$, the set of internal states x_i , $i = 1, \dots, N$, spans all of the Hilbert space \mathcal{H}_{U^∞} of fading memory functions. This provides a new class of systems which have universal computing power on time dependent inputs. In particular it provides theoretical justification for the widely used paradigm of reservoir computing with linear readout layer^{20,3,21}, by proving that such systems can be universal.

Applications. To illustrate our theory, we have chosen driven versions of three (simulated) dynamical systems. The first is the well-known logistic map²², governed by the equation

$$x(t+1) = \rho v(t)(1 - v(t)),$$

with $u(t)$ the input, $v(t) = \mathcal{S}(x(t) + u(t))$, and $\mathcal{S}(y)$ a piecewise linear saturation function, i.e.,

$$\mathcal{S}(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

to ensure that $0 \leq v(t) \leq 1$. The parameter ι can be used to scale the strength with which the system is driven. When not driven, this system becomes unstable for $\rho > 3$.

The second system is an *echo-state network* (ESN)²⁰, which can be thought of as a simplified model of a random recurrent neural network. ESNs are very successful at processing time dependent information^{3,21}. The N internal variables $x_i(t)$, $i = 1 \dots N$, evolve according to the following equation

$$x_i(t+1) = \tanh \left(\rho \sum_{j=1}^N w_{ij} x_j(t) + \iota v_i u(t) \right), \quad (11)$$

where the coefficients w_{ij} and v_i are independently chosen at random from the uniform distribution on the interval $[-1, +1]$, and the matrix w_{ij} is then rescaled such that its maximal singular value equals 1. The parameters ρ and ι , called the feedback gain and input gain respectively, can be varied. It was shown in^{20,23} that, if the maximal singular value of the weight matrix $W = [w_{ij}]$ equals one, ESN are guaranteed to have fading memory whenever $\rho < 1$. In the absence of input, they are unstable for $\rho > 1$. Because of the saturating nature of the tanh nonlinearity, driving an ESN effectively stabilizes the system dynamics²⁴. In this work we took an ESN with $N = 50$ variables. A single instantiation of the coefficients w_{ij} and v_i was used for all calculations presented below.

The last system is a driven version of the continuous-time continuous space two-dimensional Gray-Scott reaction diffusion (RD) system^{25,26}, a one-dimensional version of which was used in⁹ in a machine learning context. This system consists of two reagents, A and B , which have concentrations $a_{xy}(\tau)$ and $b_{xy}(\tau)$ at spatial coordinates (x, y) and at (continuous-valued) time τ . The reaction kinetics of the undriven system are described by the following differential equations:

$$\begin{cases} \frac{\partial a_{xy}}{\partial \tau} = d_a \nabla^2 a_{xy} - a_{xy} b_{xy}^2 + f(1.0 - a_{xy}) \\ \frac{\partial b_{xy}}{\partial \tau} = d_b \nabla^2 b_{xy} + a_{xy} b_{xy}^2 - (f+k) b_{xy}. \end{cases}$$

Reagents A and B diffuse through the system with diffusion constants d_a and d_b , respectively. A is converted into B at rate $a_{xy} b_{xy}^2$. B cannot be converted back into A . A is supplied to the system through a semipermeable membrane, one side of which is kept at a fixed (normalised) concentration of 1.0. The rate with which A is fed to the system is proportional to the concentration difference across the membrane with a rate constant f (the *feed rate*). B is eliminated from the system through a semipermeable membrane at a rate proportional to its concentration (the outside concentration is set to 0.0), with rate constant $f+k$. To drive the system with input signal $u(t)$, we first convert it to continuous time as follows: $u(\tau) = u(t)$, $tT_S \leq \tau < (t+1)T_S$, for a given value the sampling period T_S . This signal is then used to modulate the concentration of A outside the membrane:

$$\frac{\partial a_{xy}}{\partial \tau} = d_a \nabla^2 a_{xy} - a_{xy} b_{xy}^2 + f(1.0 + \iota w_{xy} u(\tau) - a_{xy})$$

with $|\iota w_{xy} u(\tau)| < 1.0$.

In order to break the spatial symmetry of the system, the modulation strength w_{xy} is randomized across the system. The concentration of A is observed at N discrete positions (25 in our experiments) by taking one sample per sampling period T_S . In our numerical experiments, we keep all parameters d_a , d_b , f , k , ι and w_{xy} constant, and explore the behavior of the systems as T_S is varied. Further details about the RD system are given in the supplementary material.

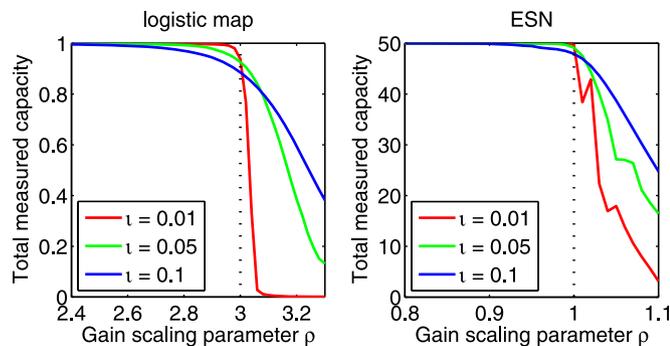


Figure 1 | Total measured capacity C_{TOT} for the logistic map (left) and an ESN with 50 nodes (right) as a function of the gain parameter ρ and for three different values of the input scaling parameter ι . The edge of stability of the undriven system is indicated in both figures by the dotted line.

Total capacity. We now apply our theoretical results to the study of these three dynamical systems. We first choose the probability distribution over the inputs $p(u)$ and the complete orthonormal set of functions y_l . We take $p(u)$ to be the uniform distribution over the interval $[-1, +1]$. As corresponding basis we use finite products of normalized Legendre polynomials for each time step:

$$y_{\{d_i\}} = \prod_i \mathcal{P}_{d_i}(u(t-i)) \quad (12)$$

where $\mathcal{P}_{d_i}(\cdot)$, $d_i \geq 0$, is the normalized Legendre polynomial of degree d_i . The normalized Legendre polynomial of degree 0 is a constant, $\mathcal{P}_0 = 1$. The index $l \equiv \{d_i\}$ therefore corresponds to the set of d_i , only a finite number of which are non zero. We do not evaluate the capacity when $\{d_i\}$ is the all zero string, i.e., at least one of the d_i must be non zero (see supplementary material for details of the expressions used). Other natural choices for the probability distributions $p(u)$ and the set of orthonormal functions (which will not be studied here) are for instance inputs drawn from a standard normal distribution and a basis of products of Hermite polynomials, or uniformly distributed inputs with a basis of products of trigonometric functions.

Each of the dynamical systems in our experiments were simulated. After an initialization period the inputs and outputs were recorded for a duration $T = 10^6$ time steps. When using finite data, one must take care not to overestimate the capacities. According to the procedure outlined in the supplementary material, we set to zero all capacities

for which $C_T(X, \{d_i\}) < \epsilon$, and therefore use in our analysis the truncated capacities $C_T^\epsilon(X, \{d_i\}) = \theta(C_T(X, \{d_i\}) - \epsilon)C_T(X, \{d_i\})$ where θ is the Heaviside function. The value of ϵ is taken to be $\epsilon = 1.7 \cdot 10^{-4}$ for ESN, $\epsilon = 1.1 \cdot 10^{-4}$ for RD system, and $\epsilon = 2.2 \cdot 10^{-5}$ for the logistic map (see supplementary material for the rationale behind these choices of ϵ). We have carefully checked that the values we obtain for $C_T^\epsilon(X, \{d_i\})$ are reliable.

According to the above theorems, for fading memory systems with linearly independent state variables, the sum of all capacities $C = \sum_{\{d_i\}} C(X, \{d_i\})$ should equal the number of independent observed variables N . To test this, in Figure 1 we plot the sum of the measured capacities

$$C_{TOT}[C] = \sum_{\{d_i\}} C_T^\epsilon(X, \{d_i\}),$$

as a function of the parameter ρ for logistic map and ESN, and for three values of the input scaling parameter ι . For very small input scaling, the capacity C_{TOT} remains close to N up to the bifurcation point of the undriven system and then drops off rapidly. As the input scaling increases, the loss of capacity becomes less abrupt, but its onset shifts to lower values. We attribute this to two effects which act in opposite ways. On the one hand, when the input strength increases, the response of the system to the input becomes more and more nonlinear, and is not well approximated by a polynomial with coefficients $< \epsilon$. As a consequence C_{TOT} underestimates the total capacity. On the other hand driving these systems on average stabilizes their dynamics²⁴, and this effect increases with the input amplitude ι . Therefore the transition to instability becomes less abrupt for larger values of ρ .

The measured capacities $C_T^\epsilon(X, \{d_i\})$ contain much additional information about how the dynamical systems process their inputs. To illustrate this, we have selected a parameter that changes the degree of nonlinearity for each system: the input scaling parameter ι for the ESN and logistic map, and the sampling period T_s for the RD system, and studied how the capacities change as a function of these parameters. In Figure 2 we show the breakdown of the total capacity according to the degree $\sum_{i=1}^{\infty} d_i$ of the individual basis functions. The figure illustrates how, by increasing ι or T_s , the information processing that takes place in these systems becomes increasingly nonlinear. It also shows that these systems process information in inequivalent ways. For instance from the figure it appears that, contrary to the other systems, the ESN is essentially incapable of computing basis functions of even degree when the input is unbiased (we attribute this to the fact that the tanh appearing in the ESN is an odd function).

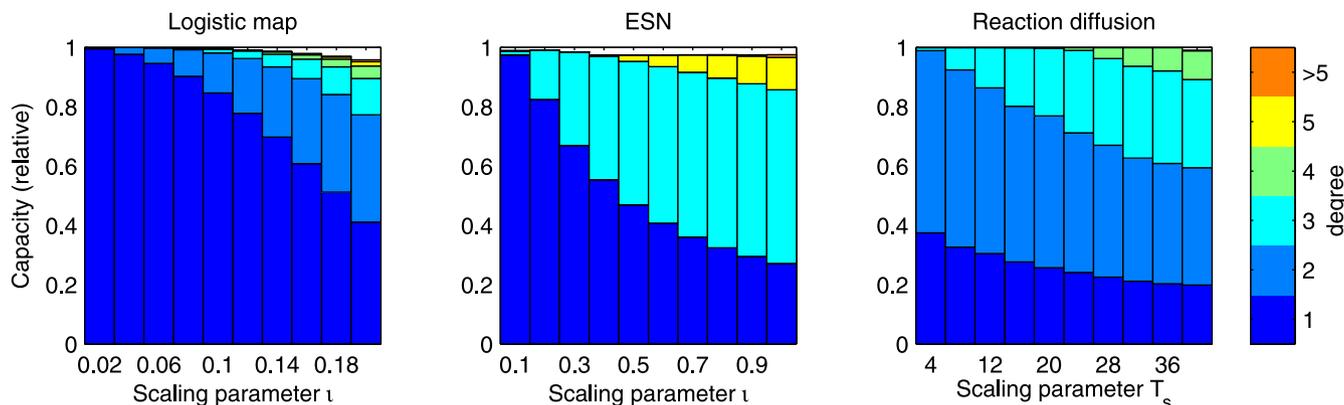


Figure 2 | Breakdown of total measured capacity according to the degree $\sum_{i=1}^{\infty} d_i$ of the basis function as a function of the parameter ι for ESN and logistic map, and T_s for RD system. The values of ρ (0.95 for ESN and 2.5 for logistic map) were chosen close to the edge of stability, a region in which the most useful processing often occurs²⁷. The scale bar indicates the degree of the polynomial. Capacities for polynomials up to degree 9 were measured, but the higher degree contributions are too small to appear in the plots. Note how, when the parameters ι and T_s increase, the systems become increasingly nonlinear. Due to the fact that the hyperbolic tangent is an odd function and the input is unbiased, the capacities for the ESN essentially vanish for even degrees.

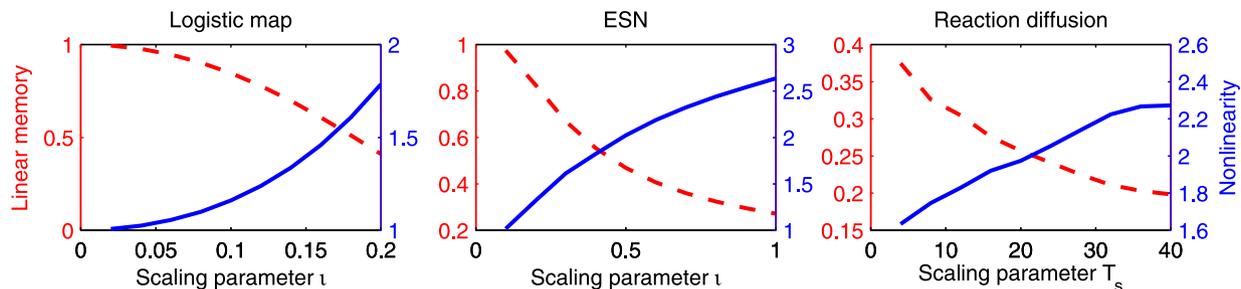


Figure 3 | Trade-off between linear memory $L[C]$ (red, dashed, left axis) and nonlinearity $NL[C]$ (blue, full line, right axis), for ESN ($\rho = 0.95$), logistic map ($\rho = 2.5$) and Reaction Diffusion systems as the parameters ι and T_s are changed.

Memory vs. non-linearity trade-off. Further insight into the different ways dynamical systems can process information is obtained by noting that the measured capacities rescaled by the number of observed signals, $C_T^c(X, \{d_i\})/N$, constitute a (possibly sub-normalised because of underestimation or failure of the fading memory property) probability distribution over the basis functions. This enables one to compute quantities which summarize how information is processed in the dynamical systems, and compare between systems. We first (following¹²) consider the *linear memory capacity* (the fraction of the total capacity associated to linear functions):

$$L[C] = \sum_{\{d_i\}} \delta \left(\sum_i d_i - 1 \right) \frac{C_T^c(X, \{d_i\})}{N}, \quad (13)$$

where the delta function is equal to 1 iff. its argument is zero, and otherwise is equal to 0. Conversely, the *non-linearity* of the dynamical system is defined by associating to each Legendre polynomial \mathcal{P}_{d_i} in eq. (12) the corresponding degree d_i , and to products of Legendre polynomials the sums of the corresponding degrees, i.e., the total degree of the basis function, to obtain

$$NL[C] = \sum_{\{d_i\}} \left(\sum_{i=1}^{\infty} d_i \right) \frac{C_T^c(X, \{d_i\})}{N}. \quad (14)$$

In Figure 3 we plot how these quantities evolve for the three dynamical systems as a function of the chosen parameters ι and T_s .

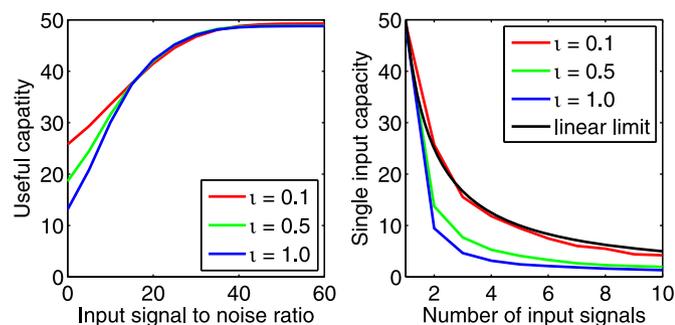


Figure 4 | Decrease of total capacity C_{TOT} due to noise, for an ESN with $\rho = 0.95$, for different values of ι , corresponding to varying degree of nonlinearity. In the left panel there are two i.i.d. inputs, the signal u and the noise v . The horizontal axis is the input signal to noise ratio in dB ($SNR = 10 \log_{10} \left(\frac{\text{var}(u)}{\text{var}(v)} \right)$). The fraction of the total capacity which is usable increases when the SNR increases, and decreases when the system becomes more non-linear (increasing ι). In the right panel there are a varying number K of i.i.d. inputs with equal power, the total power being kept constant as K varies. The capacity for computing functions of a single input in the presence of multiple inputs is plotted. The black line indicates the situation for a strictly linear system, where the capacity for a single input should equal N/K .

Unsurprisingly the linearity L and non-linearity NL are complementary, when the first decreases the second increases. We interpret this as a fundamental tradeoff in the way dynamical systems process information: on one end of the scale they store the information but then are linear, while on the other end they immediately process the input in a highly non-linear way, but then have small linear memory.

Influence of noise on the computation in dynamical systems. Real life systems are often affected by noise. The capacities enable one to quantify how the presence of noise decreases the computational power of a dynamical system. Here we model noise by assuming that there are two i.i.d. inputs u and v , representing the signal and the noise respectively. The aim is to carry out computation on u . The presence of the unwanted noise v will degrade the performances. The origin of this degradation can be understood as follows. We can define the Hilbert space $\mathcal{H}_{U\infty}$ of fading memory functions of $u^{-\infty}$, as well as the Hilbert space $\mathcal{H}_{V\infty}$ of fading memory functions of $v^{-\infty}$. The dynamical system belongs to the tensor product of these two spaces $x_i \in \mathcal{H}_{U\infty} \otimes \mathcal{H}_{V\infty}$, and a basis of the full space is given by all possible products of basis functions for each space. It is only over the full space that Theorem 7 holds and that the capacities will sum to N . However since v is unknown, the capacities which are measurable are those which depend only on u . The corresponding basis functions project onto a subspace, and therefore these capacities will not sum to N . Furthermore, we expect that the more non-linear the system, the larger the fraction of the total capacity which will lie outside the subspace $\mathcal{H}_{U\infty}$. This is illustrated in Figure 4 (left panel). We also consider in Figure 4 (right panel) the case where there are K input signals of equal strength, and the aim is to carry out computation on only one of them.

Discussion

In the present work we introduce a framework for quantifying the information processing capacity of a broad class of input driven dynamical systems, independently of their implementation. Aside from the fundamental result that under general conditions dynamical systems of the same dimensionality have globally the same computational capacity, our framework also allows testing of hypotheses about these systems.

We have already outlined several insights about the influence of system parameters on the kind of computation carried out by the dynamical system, but the potential implications go much further. While previous work on memory capacities¹²⁻¹⁵ seemed to indicate that non-linear networks tend to have bad (linear) memory properties and are thus unsuitable for long-term memory tasks, we now conclude that information is not lost inside these networks but merely transformed in various non-linear ways. This implies that non-linear mappings with memory in dynamical systems are not only realisable with linear systems providing the memory followed by a non-linear observation function, but that it is possible to find a system that inherently implements the desired mappings. Moreover,



our framework allows an a-priori evaluation of the suitability of a given system to provide a desired functional mapping, e.g., required for a class of applications.

When considering sequences that are not i.i.d. (as is the case for most real-world data), the construction of an orthonormal function basis, although it exists in theory, is often not feasible. From this perspective, the upper bound on total capacity is largely a theoretical insight. However, by using either an over-complete set of functions or an orthonormal but incomplete set, a lot of useful information about the way a system processes a given type of input data or about the type of processing required for a given task can still be gained.

An avenue for further research with high potential would be to extend the current framework such that the systems can be adapted or even constructed so as to provide certain functional mappings. Also, a study of the effects of coupling several of these systems with known computational properties on the global capacity of the whole would be very interesting in the context of larger real-world dynamical systems.

The framework is applicable under rather loose conditions and thus to a variety of dynamical systems representative of realistic physical processes. It is for instance now possible to give a complete description of the functional mappings realised by neuronal networks, as well as the influence of network structure and even homeostatic and synaptic plasticity on the realised functional mappings. The application of the proposed measures to models for different neural substrates such as the hippocampus or prefrontal cortex could further elucidate their roles in the brain. Additionally, it allows for instance to evaluate the suitability of the physical embodiment of dynamical systems such as robots to inherently perform computation, thus creating an opportunity for simplifying their controller systems. Finally, it provides a novel way to approach systems which are not generally considered to explicitly perform computation, such as intra-cellular chemical reactions or social networks, potentially yielding entirely new insights into information processing and propagation in these systems.

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Author contributions

J.D., B.S. and S.M. designed research, J.D., D.V. and S.M. performed research, all authors analyzed data and wrote the paper.

Additional information

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